Large-scale features of rotating forced turbulence

José Gaite,[1] David Hochberg,[2] and Carmen Molina-París[3]
[1]*Instituto de Matemáticas y Física Fundamental, CSIC, Serrano 123, 28006 Madrid, Spain*
[2]*Centro de Astrobiología (CSIC-INTA), Carretera de Ajalvir Km. 4, 28850 Torrejón de Ardoz, Madrid, Spain*
[3]*Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom*

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Large-scale features of a randomly isotropically forced incompressible and unbounded rotating fluid are examined in perturbation theory. At first order in both the random force amplitude and the angular velocity, we find two types of modifications to the fluid equation of motion. The first correction transforms the molecular shear viscosity into a (rotation independent) effective viscosity. The second perturbative correction leads to a new large scale nondissipative force proportional to the fluid angular velocity in the slow rotation regime. This effective force does no net work and alters the dispersion relation of inertial waves propagating in the fluid. Both dynamically generated corrections can be identified with certain components of the most general axisymmetric "viscosity tensor" for a Newtonian fluid.

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I. INTRODUCTION

The special features of turbulence in the presence of rotation have attracted the interest of many authors [1–4]. Relying on some experiments, the methods of study used have ranged from analytic approaches to numerical simulations [2,5–7]. The central theme in rotating fluids is the effect of the Coriolis force, which induces anisotropy (there is a preferred direction, that of the rotation axis). This anisotropy is extreme in the limit of fast rotation, which actually forces the flow to become two-dimensional (Proudman-Taylor theorem) [1]. In this work we apply perturbation theory to the randomly forced Navier-Stokes equation with Coriolis force as a model for the turbulent regime of a rotating fluid. The perturbative study of the ordinary randomly forced Navier-Stokes equation, in combination with the renormalization group (as an improvement of perturbation theory), has a long tradition [8–10].

The addition of the Coriolis force, induced by the rotation of the fluid, introduces one additional parameter, the angular velocity Ω or, in dimensionless form, either the Rossby or Ekman numbers (in addition to the Reynolds number) [1]. Let us focus on the Ekman number, $Ek=\nu/(\Omega L^2)$, that depends on the viscosity ν and a scale L, roughly associated with the size of the fluid system. The Ekman number gives the relative importance of the viscosity and Coriolis forces. We will *assume* henceforth that for small Ω ($<\nu/L^2$) the turbulence is isotropic and the only relevant parameter is the viscosity. In this limit, the results of the study of the ordinary randomly forced Navier-Stokes equation hold (the random force is always assumed isotropic).

For larger Ω we will encounter new features. In fact, the only restriction on perturbative correction terms is that they respect the basic *symmetry* of the equations, in our case, the axial symmetry about the rotation axis. We will see that perturbation theory generates new terms fulfilling these symmetry constraints. Therefore, one must find the complete set of allowed terms that can arise in perturbation theory. We will determine all the terms that can be represented by the components of an axisymmetric "viscosity tensor" and, in particular, those that arise at first order in perturbation theory. The part of this "viscosity tensor" that is pair antisymmetric in the indices plays a significant role; however, it does not lead to dissipation, and therefore, is not truly viscous.

As in homogeneous and isotropic turbulence, we assume that the physical region of study is sufficiently far from the surfaces, where the boundary conditions are imposed, for them not to have any *direct* effect, except the presence of the scale L. In contrast to ordinary turbulence, this condition only implies that we can have homogeneous turbulence but, due to rotation, it cannot be isotropic. It is pertinent to mention here that the possibility of anisotropic forced turbulence, and precisely with axial symmetry, has already been considered [11]. In this reference, however, the authors assume that the breakdown of isotropy occurs through a random force whose two-point correlation function depends on the anisotropy vector $\vec{n}$. They derive a renormalized force proportional to second and fourth powers of $\vec{n}$. In our case, we will see that the first perturbative correction is linear in $\vec{\Omega}$, like the Coriolis force itself.

This paper is organized as follows. In Sec. II we introduce the randomly forced hydrodynamical equations with rotation. We assume that the fluid is incompressible and show how to formulate them as a problem of homogeneous but anisotropic incompressible turbulence. Fourier analysis of the turbulent velocity field is used to organize the perturbation expansion [10] in Sec. II B. We introduce in Sec. II C the linear response function. Unlike the isotropic case, the Coriolis term leads to a *nonsymmetric* linear response function matrix. In Sec. II D we define the nonlinear response function and present its perturbative expansion (slow rotation). We also compute the first order perturbative correction to the response function, which allows the identification of the (rotation independent) effective shear viscosity (proportional to the cube of the Reynolds number), and a new anisotropic force. In Sec. III we write down the most general axially symmetric "viscosity tensor," as the existence of a preferred direction, singled out by the fluid rotation, requires the introduction and use of axisymmetric tensors. This rank four ''viscosity tensor'' expresses the proportionality between the fluid stress tensor and the rate of strain tensor. In isotropic and homogeneous incompressible turbulence, the viscosity tensor depends only on one parameter, the fluid shear viscosity. In the case of rotating turbulence, and for slow rotation, we find that the axisymmetric ''viscosity tensor'' depends on two parameters: the molecular shear viscosity (coming from the isotropic terms of the ''viscosity tensor'') and a new one, that arises from the anisotropic terms in the ''viscosity tensor.'' We also show that this new parameter can be identified as the coefficient of the anisotropic force calculated perturbatively (in the previous Sec. II D). Having thus established the equivalence between the perturbatively corrected randomly forced Navier-Stokes equation with Coriolis force on the one hand, and a (Newtonian) rotating incompressible fluid with an effective axisymmetric ''viscosity tensor'' on the other, we proceed, in Sec. IV, to discuss some physical consequences of the new terms in the perturbed fluid equations. In Sec. IV A we consider the quasilocal force induced by the anisotropic components of the ''viscosity tensor'' and show that it is proportional to the cube of the Reynolds number, and that it does not lead to dissipation. In Sec. IV B we study the dynamical effects of this force on the propagation of inertial waves. We end by discussing our results and proposing further work on the problem of rotating turbulence. In Appendix A we introduce the diagrammatic representation of the exact Navier-Stokes equation and the diagram encoding the first order correction to the response function and in Appendix B we present the technical details needed to carry out the perturbative calculation in the slow rotation limit.

II. BASIC EQUATIONS AND PERTURBATION THEORY

A. Equations of motion with random force in a rotating frame

We start from the hydrodynamical equations for a fluid with density field $\rho(\mathbf{x},t)$, velocity field $\mathbf{u}(\mathbf{x},t)$, pressure $p(\mathbf{x},t)$, and molecular shear (tangential) and bulk kinematic viscosities ν and κ, respectively. We assume that the fluid is rotating with constant angular velocity $\mathbf{\Omega}$ along the $\hat{z}$ axis and that it is subject to an isotropic random forcing *per unit mass* $\mathbf{f}$. The mass and momentum conservation equations are

$$\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot(\rho\mathbf{u})=0, \tag{1a}$$

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\vec{\nabla})\mathbf{u}=-\frac{1}{\rho}\vec{\nabla}p+\nu\nabla^2\mathbf{u}+\left[\kappa+\nu\left(\frac{d-2}{d}\right)\right]\vec{\nabla}(\vec{\nabla}\cdot\mathbf{u}) -2\mathbf{\Omega}\times\mathbf{u}-\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{x})+\mathbf{f}, \tag{1b}$$

where d is the number of space dimensions. The dimension of space d will be kept as a free variable, although when we consider rotation-dependent expressions, these must be evaluated for $d=3$. The momentum equation (1b) is supplemented with a random stirring force that leads to a statistical distribution for the velocity field and can be used to model turbulent flows just as is done for isotropic randomly stirred (nonrotating) turbulence [8,10]. Regarding the random force spectrum and statistics, we take a Gaussian random force that is white in time, for simplicity, but we allow for (translation invariant) spatial correlations. So we can write

$$\langle f_i(\vec{x},t)\rangle=0 \quad\text{and}\quad \langle f_i(\vec{x},t)f_j(\vec{x}\,',t')\rangle=D_{ij}(\vec{x}-\vec{x}\,')\,\delta(t-t'), \tag{2}$$

where the angular brackets denote an average over the random force realizations. The spectral function for $D_{ij}(\vec{x})$ will be specified below.

We assume that the fluid is incompressible so that the density field is constant $[\rho(\vec{x},t)=\rho_0]$ and $\vec{\nabla}\cdot\mathbf{u}=0$. Under this condition we need only consider the equation for the conservation of momentum (1b) and write

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\vec{\nabla})\mathbf{u}=-\frac{1}{\rho_0}\vec{\nabla}\left[p-\frac{\rho_0}{2}(\mathbf{\Omega}\times\mathbf{x})^2\right] +\nu\nabla^2\mathbf{u}-2\mathbf{\Omega}\times\mathbf{u}+\mathbf{f}. \tag{3}$$

Notice that the force per unit mass $\mathbf{f}$ will be taken solenoidal as well, that is $\vec{\nabla}\cdot\mathbf{f}=0$, in order to avoid having a random component in the pressure.

In the absence of random stirring particular solutions of Eq. (3) are well known (for an incompressible fluid): it admits plane wave solutions, called *inertial waves* [1,12,13]. These are exact solutions of the nonlinear equations, but superposition does not hold. They may have a role in the transition to turbulence [4]. In Sec. IV B we will study how the perturbative corrections modify the propagation of inertial waves.

We now proceed to eliminate the gradient term of Eq. (3) by making use of the incompressibility condition [10]. We define the generalized pressure as $p^*\equiv p-(\rho_0/2)(\mathbf{\Omega}\times\mathbf{x})^2$. By taking the divergence of the previous equation we can solve for p^* to obtain

$$p^*=-\rho_0\frac{1}{\nabla^2}[\partial_i(u_j\partial_j u_i)+2\epsilon_{ijk}\Omega_j\partial_i u_k], \tag{4}$$

so that the pressure p^* can be eliminated from Eq. (3) by writing

$$-\frac{1}{\rho_0}\vec{\nabla}p^*=\vec{\nabla}\frac{1}{\nabla^2}\vec{\nabla}\cdot[(\mathbf{u}\cdot\vec{\nabla})\mathbf{u}+2\mathbf{\Omega}\times\mathbf{u}]. \tag{5}$$

We can write the Navier-Stokes equation as follows:

$$\frac{\partial \mathbf{u}}{\partial t}+\lambda\mathcal{P}[(\mathbf{u}\cdot\vec{\nabla})\mathbf{u}]=\nu\nabla^2\mathbf{u}-\mathcal{P}(2\mathbf{\Omega}\times\mathbf{u})+\mathbf{f}, \tag{6}$$

where, following standard practice, we have introduced the constant λ in front of the advective term for book-keeping purposes [8] (λ will be useful when carrying out the perturbation expansion and is to be set to one afterwards). The projection operator $\mathcal{P}$ is given by

$$\mathcal{P}=\mathbf{1}-\vec{\nabla}\frac{1}{\nabla^2}\vec{\nabla}, \tag{7}$$

and ensures that the nonlinear and Coriolis terms are solenoidal. In Eq. (6), if **u** is solenoidal so is **f** and vice versa.

Unlike Eqs. (1b) or (3), Eq. (6) is translation invariant. That is, the centrifugal term in (1b) clearly distinguishes the origin ($\mathbf{x}=\mathbf{0}$) as a special point; but as we have seen in Eq. (5) we can include this term into the generalized pressure and eliminate p^* from the equation. This yields Eq. (6) in which a preferred *direction* (but no preferred point) is singled out by the angular velocity. This latter equation is invariant under translations, hence, we can make use of the Fourier transform. Since in Fourier space

$$\mathcal{P}_{ij}(\mathbf{k})=\delta_{ij}-\frac{k_i k_j}{k^2}, \tag{8}$$

Eq. (6) only contains vectors orthogonal to **k** and we may refer to this equation as the *transverse* Navier-Stokes equation.

We choose the random force spectrum (2) as follows:

$$\langle f_i(\vec{k},\omega)\rangle=0$$

and

$$\langle f_i(\vec{k},\omega) f_j(\vec{k}',\omega')\rangle = (2D)k^{-y}(2\pi)^{d+1}\mathcal{P}_{ij}(\mathbf{k})\,\delta(\omega+\omega')\,\delta^d(\vec{k}+\vec{k}'), \tag{9}$$

where $k=|\vec{k}|$, $D>0$ is a measure of the amplitude of the random force and the real exponent $y>-2$ characterizes the random force spectrum [9]. When $y=d$ the velocity correlations produce an energy spectrum resembling the Kolmogorov spectrum. The random force acts *isotropically* on the fluid, thus, whatever anisotropies emerge at large scale must be due to the Coriolis force. In the following sections we analyze the nature of the anisotropies by solving Eq. (6) in perturbation theory.

B. The Fourier transformed equation

In what follows we use the Fourier transformed equation of motion (6). Our convention for the Fourier transform is given by

$$u_j(\vec{x},t)=\int_{k<\Lambda}\frac{d^d\vec{k}}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}u_j(\vec{k},\omega)e^{i(\vec{k}\cdot\vec{x}-\omega t)}. \tag{10}$$

We have introduced a wave-number cutoff Λ, so that the integral over $\vec{k}$ is restricted to the values $|\vec{k}|<\Lambda$. The inverse of this cutoff, $1/\Lambda$, can be associated with the dissipation (Kolmogorov) scale, and we assume that $1/\Lambda \ll L$ [9].

In order to write the Navier-Stokes Eq. (6) in wave-number representation we transform the **u** field according to Eq. (10), apply convolution to the nonlinear term, and invert the Fourier transform to obtain

$$\begin{aligned}(-i\omega+\nu k^2)u_i(\vec{k},\omega)+\mathcal{P}_{ij}(\mathbf{k})(2\boldsymbol{\Omega}\times\mathbf{u})_j &= -\frac{i}{2}\lambda[\mathcal{P}_{ik}(\mathbf{k})k_j+\mathcal{P}_{ij}(\mathbf{k})k_k]\int_{p<\Lambda}\frac{d^d\vec{p}}{(2\pi)^d}\\ &\times\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}u_j(\vec{k}-\vec{p},\omega-\omega')u_k(\vec{p},\omega')+f_i(\vec{k},\omega).\end{aligned} \tag{11}$$

This equation, but without the Coriolis term, is a familiar expression in turbulence research [8–10]. Equation (11) can be iterated to any desired order in λ and will serve as the starting point for constructing the perturbation expansion, which is considered in the following Section.

C. The linear response function

If we "shutoff" the nonlinear terms (proportional to λ) in Eq. (11) we can identify the (inverse) linear response function, which will be used in carrying out the perturbative calculation. Thus, setting $\lambda=0$ in Eq. (11) we have

$$(-i\omega+\nu k^2)u_i^{(0)}(\vec{k},\omega)+\mathcal{P}_{ij}(\mathbf{k})(2\boldsymbol{\Omega}\times\mathbf{u}^{(0)})_j=f_i(\vec{k},\omega), \tag{12}$$

where $\mathbf{u}^{(0)}$ is the linear velocity field. The preceding equation can be written as

$$[G_0(\vec{k},\omega)]^{-1}\mathbf{u}^{(0)}(\vec{k},\omega)=\mathbf{f}(\vec{k},\omega), \tag{13}$$

which defines the linear (inverse) response function matrix, and allows one to solve for the linear velocity field $\mathbf{u}^{(0)}(\vec{k},\omega)$. In the following section we calculate the perturbative corrections to the linear inverse response function due to the existence of nonlinearities and random forcing, and establish their effect on the physical parameters (e.g., viscosity) describing the transverse Navier-Stokes equation.

D. The nonlinear response function and its first order perturbative expansion

In this section we proceed to expand Eq. (11) by iteration, in powers of the nonlinear coupling [see the diagrammatic representation, Appendix A (A1). In this way one can calculate the perturbative corrections to the response function to any desired order in λ. We do not present all the details of such expansion as these can be found in review articles [14] and textbooks [10].

The calculation of the nonlinear response function $[G]$ requires the correction matrix $[M]$ which is defined by a recursion formula

$$\begin{aligned}[G]\equiv[G_0]+[G_0][M][G]&=[G_0]+[G_0][M][G_0]\\ &+[G_0][M][G_0][M][G_0]+\cdots,\end{aligned} \tag{14}$$

which implies

$$[G]^{-1}=[G_0]^{-1}-[M]. \tag{15}$$

In Appendix A we present the diagrammatic representation of the exact transverse Navier-Stokes equation, as well as the first order correction to the response function. The diagram of Eq. (14) with the correction $[M]$, calculated to first order of perturbation theory in the random force amplitude, is presented in diagram (A2). When written in components, the correction matrix $[M]$ is given by

$$M_{mn}(\vec{k},\omega)=\left(\frac{-i\lambda}{2}\right)^2\times4\times[k_r\mathcal{P}_{mj}(\mathbf{k})+k_j\mathcal{P}_{mr}(\mathbf{k})]I_{rjn}(\vec{k},\omega), \tag{16}$$

where the factor 4 is combinatorial [see diagram (A2)], and the function $I_{rjn}(\vec{k},\omega)$ is defined by

$$I_{rjn}(\vec{k},\omega)=\int_{1/L<|\vec{p}|<\Lambda}\frac{d^d\vec{p}}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}[(k_s-p_s)\mathcal{P}_{ln}(\mathbf{k}-\mathbf{p})+(k_n-p_n)\mathcal{P}_{ls}(\mathbf{k}-\mathbf{p})]\times(2D)|\vec{p}|^{-y}[G_0(\vec{p},\omega')]_{ra}\mathcal{P}_{ab}(\mathbf{p})\times[G_0(-\vec{p},-\omega')]^T_{bs}[G_0(\vec{k}-\vec{p},\omega-\omega')]_{jl}, \tag{17}$$

(where T means the transposed matrix). The integral over $\vec{p}$ would be divergent at $\vec{p}=\vec{0}$, so it is cutoff at the low wave-number given by the inverse of the system size scale, $1/L$. The evaluation of Eq. (17) also involves an integration over the frequency ω', which has been carried out for *slow rotation* (see Appendix B).

As we are interested in the late time and large scale limits we have computed $M_{mn}(\vec{k},\omega)$ only for $\omega=0$ and up to second order in $\vec{k}$ [8]. The integration over wave-number $\vec{p}$ can be decomposed into an integration over modulus and angles. The integrand must be expanded, up to second order in $\vec{k}$, as this is sufficient to obtain the effective viscosity [see Eq. (B2)]. The calculation of $I_{rjn}(\vec{k},\omega)$ is straightforward but tedious and its technical details are presented in Appendix B.

According to Eq. (15), the matrix $[M]$ provides the order λ^2 correction to the inverse response function. To obtain $[M]$ (for $\omega=0$ and in the limit $\mathbf{k}\to\mathbf{0}$), we carry out the integral of Eq. (17) and substitute into Eq. (16):

$$M_{ij}(\vec{k},0)=-\lambda^2\frac{(2D)S_d}{(2\pi)^d(2\nu)^2}\frac{\Lambda^{d-y-4}-(1/L)^{d-y-4}}{(d-y-4)d(d+2)}\times(d^2-y-4)k^2\mathcal{P}_{ij}(\mathbf{k}) \tag{18a}$$

$$+\lambda^2\frac{(2D)S_d(2\Omega_m)}{(2\pi)^d(2\nu)^3}\frac{\Lambda^{d-y-6}-(1/L)^{d-y-6}}{(d-y-6)d(d+2)}\times(-d^2+d+2)[k_nk_j\epsilon_{nmi}-k^2\epsilon_{nmj}\mathcal{P}_{in}(\mathbf{k})], \tag{18b}$$

where S_d is the surface area of the unit sphere in d dimensions, Λ and $1/L$ are, respectively, the upper and lower wave-number cutoffs, $\lambda=1$, and ϵ_{ijk} the Levi-Civita tensor for $d=3$. In arriving at this final form we have assumed slow rotation, that is, we have kept the rotation dependent terms up to $O(\Omega)$ in the calculation. This is explained in Appendix B and we will come back to this point in Sec. V. We can compare the linear inverse response function (B1) to its first order correction $M_{ij}(\vec{k},0)$. In order to do so, we set $\omega=0$ in Eq. (B1) to obtain

$$[G_0(\vec{k},0)]^{-1}_{ij}=\nu k^2\delta_{ij}+2\epsilon_{nmj}\Omega_m\mathcal{P}_{in}(\mathbf{k}), \tag{19}$$

which is split into an isotropic part (proportional to ν) and an anisotropic part (proportional to Ω). The first part of $[M]$ (18a) corrects the molecular shear viscosity ν, exactly as in the absence of rotation [8]. The nonlinear inverse response function must take the same form as $[G_0(\vec{k},0)]^{-1}_{ij}$, with ν replaced by ν', the effective viscosity [8]. The value of ν' can be obtained by making use of Eqs. (15) and (18a), so that

$$\nu'k^2\delta_{ij}=\nu k^2\delta_{ij}+\frac{(2D)S_d}{(2\pi)^d(2\nu)^2}\frac{\Lambda^{d-y-4}-(1/L)^{d-y-4}}{(d-y-4)d(d+2)}\times(d^2-y-4)k^2\mathcal{P}_{ij}(\mathbf{k}). \tag{20}$$

If we multiply Eq. (20) by $\mathcal{P}_{mi}(\mathbf{k})$ and make use of the fact that $\mathcal{P}$ is idempotent, we obtain

$$\nu'=\nu+\frac{D}{2\nu^2}\frac{\Lambda^{d-y-4}-(1/L)^{d-y-4}}{d-y-4}\frac{(d^2-y-4)S_d}{d(d+2)(2\pi)^d}, \tag{21}$$

which shows that the isotropic term of $[M]$, Eq. (18a), renormalizes the molecular shear viscosity ν.

We must distinguish three cases, namely, $y<d-4$, $y=d-4$, and $y>d-4$. If $y=d-4$, naive perturbation theory fails. Given that $\Lambda L\gg1$, in the case $y<d-4$, the term Λ^{d-y-4} dominates over $(1/L)^{d-y-4}$, so we could take the limit $L\to\infty$ (the $\vec{p}$ integral is convergent in the lower limit). In contrast, in the case $y>d-4$, the term $(1/L)^{d-y-4}$ dominates and we instead neglect the contribution proportional to Λ^{d-y-4}. In consequence, the actual expansion parameter is $D\Lambda^{d-y-4}/\nu^3$ or DL^{y+4-d}/ν^3, according to whether $y<d-4$ or $y>d-4$, respectively. We will take $d-4<y<d^2-4$. The number DL^{y+4-d}/ν^3 can be identified with the cube of the Reynolds number, on account of the interpretation of L as the system size scale, and that the dissipation rate is proportional to the amplitude of the random force D [9].

In particular, in $d=3$,

$$\nu'=\nu-\frac{D}{2\nu^2}\frac{(1/L)^{-y-1}}{y+1}\frac{(y-5)S_3}{15(2\pi)^3}=\nu\left(1+\frac{1}{60\pi^2}\frac{5-y}{y+1}\frac{DL^{y+1}}{\nu^3}\right). \tag{22}$$

The most interesting case is $y=d=3$, which yields the Kolmogorov energy spectrum.

The second part of the correction (18b) has the same structure as the anisotropic term on the right-hand side of Eq. (19), on considering that the first term within the square brackets vanishes when it acts on u_j. However, it does not correspond to a renormalization of $\mathbf{\Omega}$, owing to the additional dependence on k, namely, the k^2 factor. In fact, this factor is adequate for a ''viscosity term'' that depends on $\mathbf{\Omega}$ (anisotropy). Therefore, it suggests the introduction of an anisotropic ''viscosity,'' which we address in the following section.

III. SYMMETRY REQUIREMENTS: THE EFFECTIVE "VISCOSITY TENSOR"

In the preceding section we have seen that the term (18b) induces no corrections to ν. In this section we give a physical interpretation of this anisotropic contribution, and show that it arises from an effective ''viscosity tensor'' for the rotating fluid with mixed symmetry in its indices. For a Newtonian fluid the linear relation between the rate of strain and stress tensors involves a rank four viscosity tensor, so that we can write [15]

$$T_{ij}=\frac{1}{2}\,\eta_{ijmn}\left(\frac{\partial u_m}{\partial x_n}+\frac{\partial u_n}{\partial x_m}\right)\equiv\eta_{ijmn}u_{mn}\,. \tag{23}$$

As both T_{ij} and u_{mn} are symmetric tensors the only symmetries of the ''viscosity tensor'' are the following: $\eta_{ijmn}=\eta_{jimn}$ and $\eta_{ijmn}=\eta_{ijnm}$. From the above symmetries we conclude that the ''viscosity tensor'' has 36 independent components (in $d=3$). We write η_{ijmn} as a sum of a pair-symmetric (S) and a pair-antisymmetric (A) part as follows

$$\begin{aligned}\eta_{ijmn}&=\frac{1}{2}(\eta_{ijmn}+\eta_{mnij})+\frac{1}{2}(\eta_{ijmn}-\eta_{mnij})\\&\equiv\eta^{S}_{ijmn}+\eta^{A}_{ijmn},\end{aligned} \tag{24}$$

so that η^S_{ijmn} has the same symmetries as η_{ijmn} plus $\eta^S_{ijmn}=\eta^S_{mnij}$, and η^A_{ijmn} has the same symmetries as η_{ijmn} plus $\eta^A_{ijmn}=-\eta^A_{mnij}$. There are 21 pair-symmetric and 15 pair-antisymmetric independent components. The presence of axial symmetry (induced by the Coriolis term) reduces the number of independent components of both η^S_{ijmn} and η^A_{ijmn}. The most general axisymmetric tensor (in $d=3$) can be constructed from Ω_i, δ_{ij}, and ϵ_{ijk} as follows

$$\begin{aligned}\eta^S_{ijmn}=&\,\alpha_1(\Omega^2)(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})+\alpha_2(\Omega^2)\delta_{ij}\delta_{mn}+\alpha_3(\Omega^2)\\&\times(\Omega_i\Omega_j\delta_{mn}+\Omega_m\Omega_n\delta_{ij})+\alpha_4(\Omega^2)(\Omega_i\Omega_m\delta_{jn}\\&+\Omega_j\Omega_m\delta_{in}+\Omega_i\Omega_n\delta_{jm}+\Omega_j\Omega_n\delta_{im})\\&+\alpha_5(\Omega^2)\Omega_i\Omega_j\Omega_m\Omega_n,\end{aligned} \tag{25a}$$

$$\begin{aligned}\eta^A_{ijmn}=&\,\beta_1(\Omega^2)\Omega_q(\epsilon_{qim}\delta_{jn}+\epsilon_{qin}\delta_{jm}+\epsilon_{qjm}\delta_{in}+\epsilon_{qjn}\delta_{im})\\&+\beta_2(\Omega^2)\Omega_q(\epsilon_{qim}\Omega_j\Omega_n+\epsilon_{qin}\Omega_j\Omega_m+\epsilon_{qjm}\Omega_i\Omega_n\\&+\epsilon_{qjn}\Omega_i\Omega_m)+\beta_3(\Omega^2)(\Omega_i\Omega_j\delta_{mn}-\Omega_m\Omega_n\delta_{ij}).\end{aligned} \tag{25b}$$

We observe that the number of independent components has been reduced from 21 to 5 for the pair-symmetric term and from 15 to 3 for the pair-antisymmetric part. The symmetry arguments used in deducing Eqs. (25a) and (25b) are analogous to those used in the theory of elasticity [16].

The coefficient functions $\alpha_i(\Omega^2)$ and $\beta_i(\Omega^2)$ can be written as a series in Ω^2:

$$\alpha_n=\sum_{r=0}^{+\infty}\alpha_{nr}(\Omega^2)^r,\quad \beta_n=\sum_{r=0}^{+\infty}\beta_{nr}(\Omega^2)^r. \tag{26}$$

In the limit of slow rotation (linear order in Ω) these coefficients reduce to their constant $(\Omega=0)$ value and, furthermore, the terms proportional to $\alpha_3,\alpha_4,\alpha_5,\beta_2$, and β_3 vanish at this order. We can then write for the ''viscosity tensor''

$$\eta^S_{ijmn}=\alpha_1(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})+\alpha_2\delta_{ij}\delta_{mn}, \tag{27a}$$

$$\eta^A_{ijmn}=\beta_1\Omega_q(\epsilon_{qim}\delta_{jn}+\epsilon_{qin}\delta_{jm}+\epsilon_{qjm}\delta_{in}+\epsilon_{qjn}\delta_{im}). \tag{27b}$$

The pair-symmetric part is isotropic and we can identify $\rho_0\kappa=\alpha_2+2\alpha_1/d$ and $\rho_0\nu=\alpha_1$, where κ and ν denote the molecular bulk and shear kinematic viscosities, respectively; the pair-antisymmetric part, proportional to the coefficient β_1, is identified below. Incompressibility means that $\kappa=0$, leaving only ν, in the absence of rotation. This is the molecular viscosity in the original Navier-Stokes equation (1b), which gets renormalized, becoming an effective (rotation independent) viscosity, as seen in the preceding Sec. II D. However, it is to be expected that perturbation theory, at sufficiently high order, generates a dependence of α_1 on Ω, such that one would be led to consider a rotation dependent viscosity. At the order we are working, the effect of rotation is to generate a rotation dependent anisotropic ''viscosity'' of the type (27b), as will be demonstrated below.

The anisotropic ''viscosity'' (27b) is associated to a stress tensor T^A_{ij}, which yields the following viscous force

$$F_i\equiv\partial_j T^A_{ij}=\partial_j(\eta^A_{ijmn}u_{mn}). \tag{28}$$

In wave-number representation this can be written as

$$F_i=-\left(\frac{1}{2}\right)k_j\eta^A_{ijmn}(k_n\delta_{mq}+k_m\delta_{nq})u_q=-k_jk_m\eta^A_{ijqm}u_q, \tag{29}$$

where we have made use of the symmetry within pairs of indices of the tensor η^A. In order to be consistent with the transverse Navier-Stokes equation, we must project Eq. (29) by means of $\mathcal{P}(\mathbf{k})$

$$\mathcal{P}_{ip}(\mathbf{k})F_p=-k_jk_m\mathcal{P}_{ip}(\mathbf{k})\eta^A_{pjqm}u_q. \tag{30}$$

We now substitute Eq. (27b) into the previous expression to obtain

$$\mathcal{P}_{ip}(\mathbf{k})F_p=-\beta_1\mathcal{P}_{ip}(\mathbf{k})k_jk_m\Omega_n \times(\epsilon_{pqn}\delta_{jm}+\epsilon_{pmn}\delta_{jq}+\epsilon_{jqn}\delta_{pm}+\epsilon_{jmn}\delta_{pq})u_q$$
$$=-\beta_1\Omega_n[k_qk_m\epsilon_{imn}+k^2\epsilon_{mqn}\mathcal{P}_{im}(\mathbf{k})]u_q$$
$$=-\beta_1\Omega_nk^2\epsilon_{mqn}\mathcal{P}_{im}(\mathbf{k})u_q, \tag{31}$$

where we have taken into account the antisymmetric properties of the Levi-Civita tensor and the fact that $\mathcal{P}_{ij}(\mathbf{k})k_i=\mathcal{P}_{ij}(\mathbf{k})k_j=0$. We now compare this equation with the first order perturbative correction of the linear inverse response function. The force given by Eq. (31) agrees identically with the force provided by the anisotropic part of the correction $M_{ij}(\vec{k},0)$, namely, the product of its expression in Eq. (18b) and u_j, once we choose the coefficient in Eq. (27b) to be

$$\beta_1\equiv-\rho_0\frac{(2D)(2S_d)}{(2\pi)^d(2\nu)^3}(-d^2+d+2)\frac{\Lambda^{d-y-6}-(1/L)^{d-y-6}}{(d-y-6)d(d+2)}. \tag{32}$$

We have thus shown that the contribution (18b) to the response function can be understood as arising from the anisotropic ‘‘viscosity’’ tensor (27b). Since we are considering the case $y>d-4$, we can neglect the term Λ^{d-y-6}. The expansion parameter, proportional to D/ν^3, must again be identified with the cube of the Reynolds number. We postpone this identification to the following section.

If we set $d=3$ in the expression for β_1, we obtain

$$\beta_1=\rho_0\frac{(2D)(2S_3)}{(2\pi)^3(2\nu)^3}4\frac{(1/L)^{-y-3}}{(y+3)15}=\frac{\rho_0}{15\pi^2(y+3)}\frac{DL^{y+3}}{\nu^3}. \tag{33}$$

IV. EFFECTIVE LARGE-SCALE DYNAMICS

In this section we focus on the new force $F_i=\partial_jT^A_{ij}$ that appears in the fluid equation of motion due to first order perturbative corrections, and discuss some possible physical implications.

A. The quasilocal force

Here we use the results derived in previous sections to ‘‘correct’’ the transverse Navier-Stokes equation with the newly generated (first order in Ω) terms that arise at large scales due to rotation. In coordinate space the force (28) becomes

$$F_i=\beta_1\{-(\vec{\Omega}\times\nabla^2\vec{u})_i+\partial_i[\vec{\Omega}\cdot(\vec{\nabla}\times\vec{u})]\}. \tag{34}$$

In deriving this equation we have made use of the incompressibility condition and the antisymmetry properties of the Levi-Civita tensor. This is a quasi-local[1] force (per unit volume) in which the angular velocity couples to the Laplacian of the fluid velocity and to the vorticity $\boldsymbol{\omega}=\vec{\nabla}\times\mathbf{u}$.

[1]Quasilocal, in general, means depending on a function (in this case the velocity field $\mathbf{u}$) and its derivatives.

The physical character of the force F_i can be revealed by writing the *effective* equation of motion in coordinate space. We have

$$\frac{\partial\mathbf{u}}{\partial t}+\lambda\mathcal{P}(\mathbf{u}\cdot\vec{\nabla}\mathbf{u})=\nu'\nabla^2\mathbf{u}-\mathcal{P}(2\boldsymbol{\Omega}\times\mathbf{u}) -\beta_1'\mathcal{P}(2\boldsymbol{\Omega}\times\nabla^2\mathbf{u})+\mathbf{f}, \tag{35}$$

where ν' is the rotation independent effective kinematic viscosity, [see Eq. (21)], and $\beta_1'=\beta_1/(2\rho_0)$. The magnitude of the correction linear in Ω (the new force $\vec{F}$) can be determined by comparing it with either the Coriolis force or the viscosity correction. We obtain for the ratio to the Coriolis force: $\beta_1'|\nabla^2\mathbf{u}|/|\mathbf{u}|\sim DL^{y+1}/\nu^3(L\Lambda)^2$, assuming that the scale of spatial velocity fluctuations is $1/\Lambda$, that is, $|\nabla^2\mathbf{u}|/|\mathbf{u}|\sim\Lambda^2$. Hence, the expansion parameter can be identified with the cube of the Reynolds number (as in the preceding section) times $(L\Lambda)^2$. On the other hand, the ratio between the correction linear in Ω and the correction linear in ν' is $\sim\Omega L^2/\nu\sim Ek^{-1}$. This was to be expected, since Ek measures the relative strength of the viscosity and Coriolis force.

We conclude this section by providing an important property of the new force. In general, for a stress tensor associated with a pair-antisymmetric ‘‘viscosity tensor,’’ the power is given by [13]

$$P\propto\int d^3\vec{x}u_{ij}T^A_{ij}=\int d^3\vec{x}u_{ij}\eta^A_{ijmn}u_{mn}, \tag{36}$$

but, as $\eta^A_{ijmn}=-\eta^A_{mnij}$, we conclude that $P=0$; therefore, this stress tensor T^A_{ij} does not lead to dissipation and is not truly viscous. This implies that the name ‘‘viscosity tensor’’ is not appropriate, and we have only introduced it by analogy with the truly viscous pair-symmetric tensor [13].

B. Inertial waves

As already mentioned, rotating incompressible fluids support wave solutions that are exact solutions of the nonlinear equations [1,12]. Although we have derived the quasilocal force from the randomly forced equations (that describe turbulence), it is of interest to see how this force affects these wave solutions. Let us consider wave solutions of the form $e^{i(\vec{k}\cdot\vec{x}+\omega t)}$ as single mode plane waves of the linearized equation (35) (the incompressibility condition annihilates the advective term) but without the forcing term

$$\frac{\partial\mathbf{u}}{\partial t}=\nu'\nabla^2\mathbf{u}-\mathcal{P}(2\boldsymbol{\Omega}\times\mathbf{u})-\beta_1'\mathcal{P}(2\boldsymbol{\Omega}\times\nabla^2\mathbf{u}). \tag{37}$$

We can write this equation in components as follows

$$(i\omega+\nu'k^2)u_i=2(-1+\beta_1'k^2)\mathcal{P}_{im}(\mathbf{k})\epsilon_{mqn}\Omega_qu_n. \tag{38}$$

We are free to choose $\boldsymbol{\Omega}=(\Omega_x,0,\Omega_z)$ and $\mathbf{k}=(0,0,k)$. This choice implies that $u_z=0$ (incompressibility condition), so that the waves are transverse. From our choice for $\mathbf{k}$ the only nonvanishing components of $\mathcal{P}(\mathbf{k})$ are $\mathcal{P}_{xx}$ and $\mathcal{P}_{yy}$, which are equal to one. We can write the x and y components [the z component of Eq. (38) is identically null)] as follows

$$(i\omega+\nu' k^2)u_x=-2(-1+\beta_1' k^2)\Omega_z u_y, \tag{39a}$$

$$(i\omega+\nu' k^2)u_y=2(-1+\beta_1' k^2)\Omega_z u_x. \tag{39b}$$

These equations yield the (complex) frequency of the plane waves (dispersion relation)

$$\omega(\vec{k})=\pm 2\Omega \cos\theta(1-\beta_1' k^2)+i\nu' k^2, \tag{40}$$

where θ is the angle between $\mathbf{k}$ and $\boldsymbol{\Omega}$, so that $\mathbf{k}\cdot\boldsymbol{\Omega}=k\Omega\cos\theta$, with $k=|\vec{k}|$ and $\Omega=|\vec{\Omega}|$. By making use of the dispersion relation (40) it is easy to see that the waves are circularly polarized and transverse

$$u_y=\pm i u_x, \quad u_z=0. \tag{41}$$

The wave packets are not solutions of the nonlinear equations and can only be considered for small wave amplitudes. The group velocity $\mathbf{V}$ of a wave packet is (for vanishing viscosity)

$$V_i(\vec{k})\equiv\frac{\partial\omega(\vec{k})}{\partial k_i}=\pm\frac{2(1-\beta_1' k^2)}{k}\mathcal{P}_{ij}(\mathbf{k})\Omega_j \mp 4\beta_1'\frac{k_i}{k}\boldsymbol{\Omega}\cdot\mathbf{k}, \tag{42}$$

and the phase velocity $\mathbf{v}$ (for vanishing viscosity)

$$v_i(\vec{k})\equiv\frac{\omega(\vec{k})}{k^2}k_i=\pm 2(1-\beta_1' k^2)\frac{\boldsymbol{\Omega}\cdot\mathbf{k}}{k^3}k_i. \tag{43}$$

The "standard textbook" result is recovered by taking the limit $\beta_1'\to 0$ [1,12]. From this calculation we see that the new term in Eq. (37) not only changes the wave frequency and phase velocity, but also the group velocity. Moreover, the group velocity is no longer perpendicular to the phase velocity

$$\mathbf{V}\cdot\mathbf{k}=\mp 4\beta_1' k\boldsymbol{\Omega}\cdot\mathbf{k}\neq 0. \tag{44}$$

The quasilocal force would cause the energy transport not to be perpendicular to the phase velocity and a small fraction of the energy in the waves to be transported parallel to the wave vector.

V. DISCUSSION

We have applied perturbation theory to a homogeneous incompressible viscous fluid subject to solid body rotation and isotropic random forcing. At small scales, we assume that only the molecular shear viscosity and the Coriolis force are needed in writing down the *transverse* Navier-Stokes equation (6). Our (first order) perturbative results demonstrate that (i) anisotropic components of the effective "viscosity tensor" are dynamically generated at large scales by the combined interplay of the Coriolis force, the random forcing term, and the inherent nonlinearity of the Navier-Stokes equation, (ii) the molecular shear viscosity ν gets corrected in the same manner as for isotropic randomly stirred turbulence [8–10].

These perturbative results are corroborated by a symmetry principle. By making use of the axial symmetry of a rotating fluid, we have constructed the most general "viscosity tensor" that is invariant under such symmetry. The preferred direction singled out by the angular velocity $\boldsymbol{\Omega}$ breaks the isotropy and leads to new terms in the "viscosity tensor" absent in the isotropic case [13]. We have also determined and described the role of the first order perturbative new term in the effective fluid equation of motion. It acts as a quasilocal force and, like the Coriolis force, is not dissipative and does no net work on the fluid. Most importantly, we find that this quasilocal force affects the propagation of inertial waves in rotating fluids. For small wave amplitudes a fraction of the wave energy is transported in the same direction as the phase velocity.

The perturbative calculation is developed as a double expansion: in both the amplitude of the random force and the angular velocity. The actual dimensionless expansion parameters turn out to be the cube of the Reynolds number and this number times the inverse of the Ekman number, respectively. For simplicity, we have restricted ourselves to the computation of the lowest order in both: that is, first order in the random force amplitude and linear order in $\boldsymbol{\Omega}$. This is sufficient to generate two, out of the total of eight tensor terms (seven, on account of incompressibility) allowed by axial symmetry [see Eqs. (25a) and (25b)]. We conjecture that, at first order in the random force amplitude, all the remaining tensor terms are generated for higher powers in Ω: α_3, α_4, and β_3 at quadratic order, β_2 at cubic order, and finally α_5 at quartic order. Of course, as increasing powers of Ω are taken into account, the coefficient functions (26) must be expanded out to the order of Ω being investigated. The higher order terms will be needed to study the effects of fast rotation and to track the onset of the bidimensionalization of the fluid [4].

An important step in this direction will be provided by a complete renormalization group (RG) analysis of the large-scales properties of a rotating incompressible fluid. In order to carry out this RG analysis one may need to calculate the (perturbative) corrections to the nonlinear coupling term λ, as well as the random force amplitude D, and combine these two with the response function calculation presented in this paper. Once in hand, the RG fixed points can be determined and the corresponding asymptotic behavior of the rotating fluid deduced, allowing us to compute quantities such as the scale-dependent Reynolds and Ekman numbers, among others. We hope to report on these developments elsewhere.

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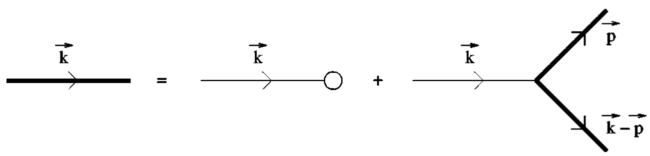
APPENDIX A: DIAGRAMMATIC REPRESENTATION OF THE EXACT EQUATION AND THE FIRST ORDER CORRECTION TO THE RESPONSE FUNCTION

In this section we include a diagrammatic representation of the exact transverse Navier-Stokes equation [see Eq. (11)] and of the recursion relation for the first order response function [see Eq. (14)]. We have followed the convention introduced in Ref. [8]. The exact equation can be represented as

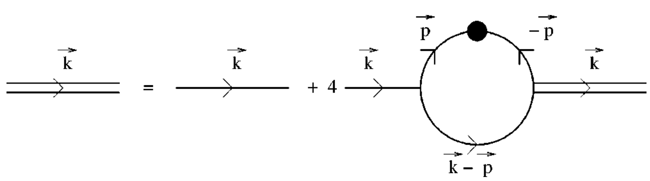


(A1)

(Exact hydrodynamic equation)

where the thick lines represent $\mathbf{u}(\vec{k},\omega)$, the open circles represent $f(\vec{k},\omega)$, and the thin lines represent $G_0(\vec{k},\omega)$. This graphic representation implies that the linear velocity field $\mathbf{u}^{(0)}(\vec{k},\omega)$ is given by a thin line attached to an open circle, as we have $\mathbf{u}^{(0)}(\vec{k},\omega)=G_0(\vec{k},\omega)\mathbf{f}(\vec{k},\omega)$.

In this diagrammatic representation we can also depict the recursion relation for the response function $G_{ij}(\vec{k},\omega)$, Eq. (14), given by the first order correction $[M]$, [Eq. (16)], as follows



(A2)

(Recursion relation for the first order response function)

where the solid circle represents the random force average $\langle f_m(\vec{p},\omega')f_n(-\vec{p},-\omega')\rangle$ introduced in Eq. (9), and the double thin lines represent the first order response function $[G]$.

APPENDIX B: PERTURBATION EXPANSION AND EVALUATION OF THE INTEGRALS

The large distance and long time renormalizability of the transverse Navier-Stokes equation (6) implies that the corrected response function G must have the same mathematical structure as its linear counterpart G_0. G has, therefore, the same frequency and wave-number dependence as G_0 and contains the same number of parameters. The renormalization of the response function yields the (rotation independent) effective viscosity ν'. In this Appendix we outline the major points in the calculation of the first order correction to the response function [8].

Equation (13) defines the linear inverse response function, which is given by

$$[G_0(\vec{k},\omega)]^{-1}=\begin{pmatrix} a-b & h & 0 \\ f & a+b & 0 \\ c & d & a \end{pmatrix}, \tag{B1}$$

where the matrix entries are: $a=-i\omega+\nu k^2$, $b=2\Omega k_1k_2/k^2$, $c=-2\Omega k_2k_3/k^2$, $d=2\Omega k_1k_3/k^2$, $f=2\Omega(1-k_2^2/k^2)$, and $h=-2\Omega(1-k_1^2/k^2)$. In the limit of vanishing rotation $(\Omega=0)$ we recover an isotropic diagonal matrix for the linear response function and its inverse [8]

$$[G_0(\vec{k},\omega)]_{ij}^{-1}=(-i\omega+\nu k^2)\delta_{ij},$$

$$[G_0(\vec{k},\omega)]_{ij}=(-i\omega+\nu k^2)^{-1}\delta_{ij}. \tag{B2}$$

Given the matrix form of the inverse linear response function (B1) we can write for the linear response function[2]

$$[G_0(\vec{k},\omega)] = \frac{1}{a^3-ab^2-afh} \times \begin{pmatrix} a^2+ab & -ah & 0 \\ -af & a^2-ab & 0 \\ -ac-bc+df & -ad+bd+ch & a^2-b^2-fh \end{pmatrix}. \tag{B3}$$

The first order correction to G is given in Sec. II D [see Eqs. (14)–(17)]. We have restricted ourselves to the limit of slow rotation (linear order in Ω), and therefore need to expand both G_0 and G_0^{-1} to this order. We point out that G_0^{-1} is already linear in Ω [see Eq. (B1)]. To linear order in Ω the linear response function is given by

$$[G_0(\vec{k},\omega)]=\frac{1}{a^2}\begin{pmatrix} a+b & -h & 0 \\ -f & a-b & 0 \\ -c & -d & a \end{pmatrix}+O(\Omega^2)$$
$$=\frac{1}{a}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}+\frac{1}{a^2}\begin{pmatrix} b & -h & 0 \\ -f & -b & 0 \\ -c & -d & 0 \end{pmatrix} + O(\Omega^2). \tag{B4}$$

The first step of this calculation is to carry out the frequency integration (over ω') in Eq. (17). From Eq. (19) we see that the viscosity ν is the coefficient of the k^2 term, and to obtain the effective viscosity we can set the frequency ω to zero from the outset. Once we make $\omega=0$ the integral over ω' can be computed by means of the calculus of residues [8]. We close the contour in the lower half plane, where there are in general three simple poles, which coalesce into one double pole in the limit of slow rotation. We obtain (keeping up to linear terms in Ω):

[2]This is also sometimes called the linear propagator [8,10].

$$\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}[G_0(\vec{p},\omega')]_{rt}[G_0(-\vec{p},-\omega')]_{sq}[G_0(\vec{k}-\vec{p},-\omega')]_{jl}$$

$$=\delta_{rt}\delta_{sq}\delta_{jl}\frac{\left(1+\frac{\vec{p}\cdot\vec{k}}{p^2}\right)}{(2\nu p^2)^2}+\delta_{rt}\delta_{sq}\frac{\left(1+\frac{2\vec{p}\cdot\vec{k}}{p^2}\right)}{(2\nu p^2)^3}[\Omega(\vec{k}-\vec{p})]_{jl}+\delta_{rt}\delta_{jl}\frac{\left(1+\frac{\vec{p}\cdot\vec{k}}{p^2}\right)}{(2\nu p^2)^3}[\Omega(-\vec{p})]_{sq}$$

$$+\delta_{sq}\delta_{jl}\frac{\left(2+\frac{3\vec{p}\cdot\vec{k}}{p^2}\right)}{(2\nu p^2)^3}[\Omega(\vec{p})]_{rt}+O(k^2). \tag{B5}$$

We have carried out an expansion in $k=|\vec{k}|$ and only kept up to linear order, as Eq. (16) is already linear in k, and we only need to compute the k^2 term. The previous equation was evaluated assuming a positive molecular shear viscosity coefficient ($\nu>0$). A change in sign will change the location of the poles and modify the frequency integration. We have introduced the matrix $[\Omega(\vec{p})]$ defined as follows:

$$[\Omega(\vec{p})]_{mk}=-2\epsilon_{ijk}\Omega_j\mathcal{P}_{im}(\mathbf{p}), \tag{B6}$$

which is already linear in the angular velocity.

If we make use of this intermediate result (B5) and substitute it into Eq. (17), we are left with the integration over the wave-number $\vec{p}$. From Eqs. (15) and (19) we can see that the renormalization of the molecular viscosity ν requires that we expand the first order correction $[M]$ up to second order in the wave-number $\vec{k}$. The factor in square brackets in Eq. (16) is already linear in k, so that the integral (17) only needs to be expanded to first order in k. It is important to note that this integral depends on $\vec{k}$ not only through the integrand but also through its limits of integration. This is because all wave numbers appearing in (17) must remain within the set of wave numbers to be integrated over (in this case we must ensure that both $\vec{p}$ and $\vec{k}-\vec{p}$ belong to this set) [8]. This means we must integrate over the *intersection* of the domains $1/L\leqslant|\vec{p}|\leqslant\Lambda$ and $1/L\leqslant|\vec{k}-\vec{p}|\leqslant\Lambda$, where $1/L$ is the lower cutoff. To first order in $\vec{k}$, the second inequality can be written as $1/L+k\cos\theta<p<\Lambda+k\cos\theta$, where θ is the angle between $\vec{k}$ and $\vec{p}$ ($\vec{k}\cdot\vec{p}=kp\cos\theta$). There are two cases to consider: (i) if $\cos\theta>0$ the intersection of the two intervals can be expressed as the difference of intervals $[1/L,\Lambda]-[1/L,1/L+k\cos\theta]$ and (ii) if $\cos\theta<0$ the intersection can be written as $[1/L,\Lambda]-[\Lambda+k\cos\theta,\Lambda]$. This means that the complete wave-number integration, valid up to $O(k^2)$, can be written as

$$\int_{1/L<|\vec{p}|<\Lambda,1/L<|\vec{k}-\vec{p}|<\Lambda}\frac{d^d\vec{p}}{(2\pi)^d}$$

$$=\int d\Omega_d\left(\int_{1/L}^{\Lambda}-\int_{1/L}^{1/L+k\cos\theta}-\int_{\Lambda+k\cos\theta}^{\Lambda}\right)\frac{dp\,p^{d-1}}{(2\pi)^d}+O(k^2), \tag{B7}$$

where $d\Omega_d$ is the surface element of the unit sphere in d dimensions.

We also present the following projection operator product expansions that prove to be useful when handling the intermediate steps of the computation

$$\mathcal{P}_{a,bc}(\mathbf{k}-\mathbf{p})\mathcal{P}_{dc}(\mathbf{p})=(k_a-p_a)\left(\delta_{bd}-\frac{p_bp_d}{p^2}\right)-\frac{p_ap_b}{p^2}\left(k_d-\frac{\vec{p}\cdot\vec{k}p_d}{p^2}\right)+O(k^2), \tag{B8}$$

$$\mathcal{P}_{a,bc}(\mathbf{k}-\mathbf{p})\mathcal{P}_{ad}(\mathbf{p})=k_a\mathcal{P}_{bc}(\mathbf{p})\mathcal{P}_{ad}(\mathbf{p})+O(k^2),$$

where

$$\mathcal{P}_{a,bc}(\mathbf{k})=k_b\mathcal{P}_{ac}(\mathbf{k})+k_c\mathcal{P}_{ab}(\mathbf{k}). \tag{B9}$$

Finally, various identities needed for the angular integrations are collected here. Let S_d represent the surface area of the unit d sphere and $\hat{n}_j$ denote a unit vector in the jth direction. The only angular integrations required are of the following types

$$\int d\Omega_d=S_d, \tag{B10}$$

$$\int d\Omega_d\hat{n}_i\hat{n}_j=\frac{S_d}{d}\delta_{ij},$$

$$\int d\Omega_d\hat{n}_i\hat{n}_j\hat{n}_n\hat{n}_m=\frac{S_d}{d(d+2)}(\delta_{ij}\delta_{mn}+\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}).$$

The angular integration of the product of an odd number of unit vectors over the unit d-sphere vanishes identically. If we make use of the previous results, we obtain the first order correction as written in Eqs. (18a) and (18b).

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